

ARC-QUASIANALYTIC FUNCTIONS

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ABSTRACT. We work with quasianalytic classes of functions. Consider a real-valued function $y = f(x)$ on an open subset U of \mathbb{R}^n , which satisfies a quasianalytic equation $G(x, y) = 0$. We prove that f is *arc-quasianalytic* (i.e., its restriction to every quasianalytic arc is quasianalytic) if and only if f becomes quasianalytic after (a locally finite covering of U by) finite sequences of local blowings-up. This generalizes a theorem of the first two authors on arc-analytic functions.

1. INTRODUCTION

Arc-analytic functions are functions that are analytic along every analytic arc. Arc-analytic functions were introduced by K. Kurdyka to study the geometry of *arc-symmetric* sets [6]. The first two authors proved that a real-valued function f on a smooth real-analytic variety U is arc-analytic and has subanalytic graph if and only if f becomes analytic after a locally finite covering of U by finite sequences of local blowings-up [1, Thm. 1.4]. The latter has become a basic tool in real-analytic geometry (see, for example, [7], [8] and other articles referenced therein). This paper deals with a natural generalization of arc-analytic functions to *quasianalytic classes*, and establishes the analogue of [1, Thm. 1.4] for such *arc-quasianalytic* functions; see Theorem 1.2 below.

A *quasianalytic class* associates, to every open subset $U \subset \mathbb{R}^n$, a subring $\mathcal{Q}(U)$ of $\mathcal{C}^\infty(U)$ which satisfies the basic properties of \mathcal{C}^∞ functions together with the property that the Taylor series homomorphism at any point of U is injective. See Section 2 for a precise definition. Examples are quasianalytic Denjoy-Carleman classes that are closed under differentiation (see [3]), and the class of \mathcal{C}^∞ functions which are definable in a given polynomially bounded o -minimal structure [10]. The former play an important part in analysis; in particular, in the study of certain partial differential equations.

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Quasianalytic classes are in general much bigger than the class of real-analytic functions, but nevertheless enjoy many properties of analytic functions. In [2], [3], the first two authors proved resolution of singularities for finitely-generated ideals in quasianalytic classes. The latter can be used to show that *quasianalytic sets* (i.e., sets defined by finitely many functions in a quasianalytic class) have geometric properties similar to those of analytic sets. In particular, the corresponding Zariski topology is Noetherian [3], though it seems unknown (and doubtful) whether rings of germs of quasianalytic functions are in general Noetherian. We can show nevertheless that resolution of singularities holds even for quasianalytic ideals that are not necessarily finitely generated; see Theorem 3.1, which is an important tool in the proof of our main theorem following.

Let \mathcal{Q} denote a given quasianalytic class (Section 2).

Definition 1.1. Let W be an open subset of \mathbb{R}^n . A function $f : W \rightarrow \mathbb{R}$ is called *arc-quasianalytic* if $f \circ \gamma \in \mathcal{Q}((-\epsilon, \epsilon))$, for every *quasianalytic arc* $\gamma : (-\epsilon, \epsilon) \rightarrow W$ (the latter means that $\gamma = (\gamma_1, \dots, \gamma_n)$, where each $\gamma_i \in \mathcal{Q}((-\epsilon, \epsilon))$).

We will say that a family of quasianalytic mappings $\{\pi_j : U_j \rightarrow U\}$ is a *locally finite covering* of U if (1) the images $\pi_j(U_j)$ are subordinate to a locally finite covering of U by open subsets; (2) if K is a compact subset of U , then there are compact subsets K_j of U_j , for each j , such that $K = \cup \pi_j(K_j)$ (the union is finite, by (1)).

A *modification* will denote a finite composite of *admissible local blowings-up*. (A *local blowing-up* of U is a blowing-up over an open subset of U . A (local) blowing-up is *admissible* if its centre is smooth and normal crossings with the exceptional divisor.)

Theorem 1.2. Let $f : U \rightarrow \mathbb{R}$ denote a function on a connected open set $U \subset \mathbb{R}^n$. Assume there is a nonzero quasianalytic function $G : U \times \mathbb{R} \rightarrow \mathbb{R}$ such that $G(x, f(x)) \equiv 0$. Then f is arc-quasianalytic if and only if there exists a locally finite covering $\{\pi_j : U_j \rightarrow U\}$ such that, for each j ,

- (1) π_j is a modification;
- (2) $f \circ \pi_j$ is quasianalytic.

The proof of [1, Thm. 1.4] relies on a generalization of Hensel's lemma to several variables and thus makes use of the Weierstrass preparation theorem. The latter does not hold in quasianalytic classes in general [5]. We were therefore forced to imagine a rather different proof of Theorem 1.2, involving a more technical iterative argument.

K. Nowak has used Theorem 1.2 to prove an interesting result on hyperbolic polynomials with quasianalytic coefficients and normal-crossings discriminant [9].

2. QUASIANALYTIC CLASSES

We follow the axiomatic framework of [3]. Consider a class of functions \mathcal{Q} given by the association, to every open subset $U \subset \mathbb{R}^n$, of a subalgebra $\mathcal{Q}(U)$ of $\mathcal{C}^\infty(U)$ containing the polynomial functions and stable under composition with a \mathcal{Q} -mapping (i.e., a mapping whose components belong to \mathcal{Q}). We say that \mathcal{Q} is *quasianalytic* if it satisfies the following three axioms:

- (1) *Closure under division by a coordinate.* If $f \in \mathcal{Q}(U)$ and

$$f(x_1, \dots, x_{i-1}, a, x_{i+1}, \dots, x_n) = 0,$$

where $a \in \mathbb{R}$, then $f(x) = (x_i - a)h(x)$, where $h \in \mathcal{Q}(U)$.

- (2) *Closure under inverse.* Let $\varphi : U \rightarrow V$ denote a \mathcal{Q} -mapping between open subsets U, V of \mathbb{R}^n . Let $a \in U$ and suppose that the Jacobian matrix

$$\frac{\partial \varphi}{\partial x}(a) := \frac{\partial(\varphi_1, \dots, \varphi_n)}{\partial(x_1, \dots, x_n)}(a)$$

is invertible. Then there are neighbourhoods U' of a and V' of $b := \varphi(a)$, and a \mathcal{Q} -mapping $\psi : V' \rightarrow U'$ such that $\psi(b) = a$ and $\psi \circ \varphi$ is the identity mapping of U' .

- (3) *Quasianalyticity.* If $f \in \mathcal{Q}(U)$ has Taylor expansion zero at $a \in U$, then f is identically zero near a .

Remark 2.1. Since the class \mathcal{Q} is closed under composition, axiom (1) implies that, if $f \in \mathcal{Q}(U)$, then all partial derivatives of f belong to $\mathcal{Q}(U)$. Axiom (2) implies that the implicit function theorem holds for functions of class \mathcal{Q} .

Throughout the paper, we work with a fixed quasianalytic class \mathcal{Q} . The elements of \mathcal{Q} will be called *quasianalytic functions*. A category of manifolds and mappings of class \mathcal{Q} can be defined in a standard way. The category of \mathcal{Q} -manifolds is closed under blowing up with centre a \mathcal{Q} -submanifold [3].

3. RESOLUTION OF SINGULARITIES

Resolution of singularities of a finitely generated ideal in a quasianalytic class \mathcal{Q} was proved in [2], [3]. The proof does not in fact require that the ideal be finite.

Theorem 3.1 (resolution of singularities in a quasianalytic class). *Let U be an open subset of \mathbb{R}^n (or a \mathcal{Q} -manifold), and let \mathcal{I} denote a sheaf of ideals of quasianalytic functions on U . Assume that each point of U admits a neighbourhood U' such that $\mathcal{I}|_{U'}$ is generated by a family of sections of \mathcal{I} over U' (not necessarily finite). Let K be a compact subset of U . Then there is a neighbourhood W of K in U , and a mapping $\pi : W' \rightarrow W$ given by a composite of finitely many admissible blowings-up (so W' is a \mathcal{Q} -manifold), such that the pull-back $\pi^*\mathcal{I}$ is a simple normal-crossings divisor.*

Let \mathcal{Q}_U denote the sheaf of germs of functions of class \mathcal{Q} on U . When an ideal sheaf $\mathcal{I} \subset \mathcal{Q}_U$ satisfies the assumption in Theorem 3.1, one says that $\mathcal{Q}_U/\mathcal{I}$ is *quasicoherent*.

Remark 3.2 (proof of Theorem 3.1). Every step of the proof of desingularization of an ideal (as presented in [4], for example) goes over to the case that $\mathcal{Q}_U/\mathcal{I}$ is quasicoherent, with essentially no change. We recall that the desingularization algorithm is based on resolution of singularities of a *marked ideal* given locally by \mathcal{I} together with its maximum order d . The *order* $\text{ord}_a \mathcal{I}$ of \mathcal{I} at $a \in U$ is the minimum order of elements of \mathcal{I}_a . Clearly, $\text{ord} \mathcal{I}$ is upper-semicontinuous in the quasianalytic Zariski topology. Resolution of singularities of the marked ideal (\mathcal{I}, d) involves recursively constructing and resolving associated marked ideals on local *maximal contact* subspaces of increasing codimension. The only place where the proof is not obviously the same as in the case of a finitely-generated ideal is in [4, Step II, p. 628] — factorization of \mathcal{I} as the product $\mathcal{M}(\mathcal{I}) \cdot \mathcal{R}(\mathcal{I})$ of its *monomial part* $\mathcal{M}(\mathcal{I})$ and *residual* (or *nonmonomial*) *part* $\mathcal{R}(\mathcal{I})$. We need to show that if $\mathcal{Q}_U/\mathcal{I}$ is quasicoherent, then $\mathcal{Q}_U/\mathcal{R}(\mathcal{I})$ is quasicoherent; this is a simple consequence of the division axiom (1) in Section 2.

4. PRELIMINARY LEMMAS

The lemmas of this section are needed for our proof of Theorem 1.2. We use \mathbb{N} to denote the nonnegative integers.

Lemma 4.1. *Let $f : U \rightarrow \mathbb{R}$ be an arc-quasianalytic function, where $U \subset \mathbb{R}^n$ is a connected open set. Assume that $G(x, f(x)) \equiv 0$, where G is a quasianalytic function that is not identically zero. Then f is continuous.*

Proof. Let Γ denote the graph of f . We have to show that, for any $x_0 \in U$, $(x_0, f(x_0))$ is the unique limit point of Γ (including ∞) over x_0 . Given a finite limit point p of Γ over x_0 , it follows from resolution of singularities of G that there is a quasianalytic arc $\gamma(t)$, $t \in (-\epsilon, \epsilon)$, such that $\gamma(0) = p$ and $\gamma(t)$ is a smooth point of Γ , for any $t \neq 0$. (See, for example, [3, Thm. 6.2].) Since f is arc-quasianalytic, f is continuous on every quasianalytic arc (in particular, on the projection of γ to U), so that $p = (x_0, f(x_0))$. On the other hand, suppose that Γ has an infinite limit point over x_0 . Consider sequences $\{a_n\}$ and $\{b_n\}$ in Γ tending to $(x_0, f(x_0))$ and ∞ . Then the straight lines joining the projections of a_n, b_n lift to quasianalytic curves in Γ , providing additional finite limit points over x_0 as n tends to ∞ (a contradiction). \square

Lemma 4.2. *Let U be a neighbourhood of the origin in \mathbb{R}^n , and let $c : U \rightarrow \mathbb{R}$ be a quasianalytic function. Assume that $|c(x)| \leq C|x^\alpha|$, for x near zero in the nonnegative quadrant $\{x_1 \geq 0, \dots, x_n \geq 0\}$, where $\alpha \in \mathbb{N}^n$ and C is a positive constant. Then the function $x^{-\alpha}c(x)$ is quasianalytic.*

Proof. Take i such that $\alpha_i \neq 0$. It is enough to show that c is divisible by x_i , because we can iterate the argument α_i times, for each i . Since

$|c(x)| \leq C|x^\alpha|$ in the nonnegative quadrant, and $\alpha_i \neq 0$, the function c is identically zero on the hyperplane $x_i = 0$. By axiom (1), this implies that c is divisible by x_i . \square

Given $x \in \mathbb{R}$ and $k \in \mathbb{N} \setminus \{0, 1\}$, we write $x^{1/k}$ for the positive k^{th} root of $|x|$. (This somewhat unusual convention is convenient for Lemma 4.3 following, and will intervene again in Section 5 only in the same way, for the purpose of applying Lemma 4.3.) Let \mathbb{Q}_+ denote the set of nonnegative rational numbers. Given a neighbourhood U of the origin and an n -tuple $p \in \{0, 1\}^n$ we set

$$U^p := \{x \in U : (-1)^{p_1}x_1 \geq 0, \dots, (-1)^{p_n}x_n \geq 0\}.$$

Lemma 4.3. *Let $f : U \rightarrow \mathbb{R}$ be a continuous arc-quasianalytic function in a neighbourhood of $0 \in \mathbb{R}^n$, and let $g(x) := x^{-\alpha}f(x)$, where $\alpha \in \mathbb{Q}_+^n$. Assume that $|g| \leq M$ near 0, for some $M > 0$, and that, for any $p \in \{0, 1\}^n$, there is a quasianalytic function $G = G_p : U^p \times (-2M, 2M) \rightarrow \mathbb{R}$ such that G is not identically zero and $G(x, g(x)) \equiv 0$, for any $x \in U^p$ near 0 with $x^\alpha \neq 0$. Then $|g|$ extends to a continuous function in a neighbourhood of 0.*

Moreover, if $\alpha \in \mathbb{N}^n$, then g extends to a continuous function in a neighbourhood of 0.

Proof. It is enough to show that $|g|$ or g extends continuously to 0 since the argument will apply to every x in a neighbourhood of 0. Consider a connected component C of $\mathbb{R}^n \setminus \{x^\alpha = 0\}$. We first show that $g|_C$ extends continuously to 0, by contradiction. Suppose that $g|_C$ has two distinct asymptotic values a and b at 0. Since C is locally connected at 0 and f is continuous, all points of the interval (a, b) are also asymptotic values of $g|_C$. Therefore, the function $y \mapsto G(0, y)$ is identically zero on (a, b) . Since $G(0, y)$ is quasianalytic, $G(0, y)$ is identically zero, by axiom (3); a contradiction.

For each connected component C of $\mathbb{R}^n \setminus \{x^\alpha = 0\}$, there thus exists precisely one asymptotic value λ_C of g at 0. We have to prove that the $|\lambda_C|$ coincide. Indeed, by axiom (1), since f is arc-quasianalytic and $\alpha \in \mathbb{Q}^n$, there is a positive integer u such that the composite of g^u with a quasianalytic arc γ is a quasianalytic function, and therefore continuous at 0. This means that $|g|$ extends continuously along every segment passing from one connected component of $\mathbb{R}^n \setminus \{x^\alpha = 0\}$ to another (since $|g| = (g^u)^{1/u}$). Consequently, the respective asymptotic values of $|g|$ must match.

Assume, moreover, that $\alpha \in \mathbb{N}^n$. Since f is arc-quasianalytic, g extends to a quasianalytic function on every segment. For the same reason as above, therefore, g is continuous. \square

Lemma 4.5 following is a reformulation of the basic idea of [1].

Definition 4.4. Let U denote an open subset of $\mathbb{R}^n \times \mathbb{R}$. A function $G(x, y) \in \mathcal{Q}(U)$ is *y-regular of order k* at $(x_0, y_0) \in U$ if $(\partial^k G / \partial y^k)(x_0, y_0) \neq 0$, and $(\partial^j G / \partial y^j)(x_0, y_0) = 0$ for $j < k$.

In particular, G is *y-regular of order 0* at (x_0, y_0) if $G(x_0, y_0) \neq 0$.

Lemma 4.5. *Let U be an open neighbourhood of $(0,0)$ in $\mathbb{R}^n \times \mathbb{R}$, and let $G : U \rightarrow \mathbb{R}$ denote a quasianalytic function which is y -regular of order $d > 1$ at $(0,0)$. Set $c_i(x) := (\partial^i G / \partial y^i)(x, 0)$, $0 \leq i \leq d$. Assume that $c_{d-1} \equiv 0$ and that $c_i(x) = x^{\alpha(d-i)} c_i^*(x)$ (i.e., $c_i(x)$ is divisible by $x^{\alpha(d-i)}$), for $0 \leq i < d-1$, where $\alpha \in \mathbb{N}^n$ and where $c_k^*(0) \neq 0$ for some $k < d-1$. Then*

$$H(x, y) := x^{-\alpha d} G(x, x^\alpha y)$$

is a quasianalytic function which is y -regular of order at most k at $(0,0)$. Moreover, if $\alpha \neq 0$, then $H(x, y)$ is quasianalytic and y -regular of order at most $d-1$ at $(0, y_0)$, for any $y_0 \in \mathbb{R}$ (not necessarily near 0).

Proof. Given x in a neighbourhood of 0, Taylor's formula for the function $y \mapsto G(x, y)$ at 0 gives

$$G(x, x^\alpha y) = \sum_{i=0}^{d-1} c_i(x) x^{\alpha i} \frac{y^i}{i!} + c_d(x) x^{\alpha d} y^d \rho(x, x^\alpha y),$$

where ρ is a C^∞ function such that $\rho(0,0) \neq 0$. Clearly, if $\alpha \neq 0$, then the latter equation holds for y in a neighbourhood of any fixed y_0 in \mathbb{R} , and x close enough to zero.

Since each $c_i(x) = x^{\alpha(d-i)} c_i^*(x)$, we get

$$(4.1) \quad G(x, x^\alpha y) = \sum_{i=0}^{d-1} x^{\alpha d} c_i^*(x) \frac{y^i}{i!} + x^{\alpha d} c_d(x) y^d \rho(x, x^\alpha y).$$

Thus $G(x, x^\alpha y)$ is divisible by $x^{\alpha d}$, and

$$(4.2) \quad H(x, y) = \sum_{i=0}^{d-1} c_i^*(x) \frac{y^i}{i!} + c_d(x) y^d \rho(x, x^\alpha y)$$

is a quasianalytic function (by axiom (1)).

Let k denote the smallest integer such that $c_k^*(0) \neq 0$. We will show that H is y -regular of order k at $(0,0)$, and, if $\alpha \neq 0$, then H is y -regular of order at most $d-1$ at any $(0, y_0)$. From (4.2), we get

$$(4.3) \quad \frac{\partial^k H}{\partial y^k}(x, y) = c_k^*(x) + \sum_{i=k+1}^{d-1} c_i^*(x) \frac{y^{i-k}}{(i-k)!} + c_d(x) y^{d-k} \sigma(x, x^\alpha y),$$

where σ is a C^∞ function. Thus,

$$(4.4) \quad \frac{\partial^k H}{\partial y^k}(0, 0) = c_k^*(0) \neq 0;$$

i.e., H is y -regular of order k at $(0,0)$.

Now suppose that $\alpha \neq 0$ and consider some nonzero $y_0 \in \mathbb{R}$. Since $c_{d-1} \equiv 0$, (4.2) also gives

$$(4.5) \quad \frac{\partial^{d-1} H}{\partial y^{d-1}}(x, y) = c_d(x) y \tau(x, x^\alpha y),$$

for (x, y) near enough to $(0, y_0)$, where τ is a \mathcal{C}^∞ function such that $\tau(0, 0) \neq 0$. Since $c_d(0) \neq 0$, the right-hand side of (4.5) is clearly nonzero at $(0, y_0) \neq (0, 0)$, as required. \square

The following lemma is a variation of Lemma 4.5 that will provide an alternative argument when the hypotheses of Lemma 4.5 are not satisfied. We drop the assumption that $c_{d-1} \equiv 0$, but can conclude that H is y -regular of order $< d$ only near $(0, 0)$.

Lemma 4.6. *Let $G(x, y)$ be a quasianalytic function which is y -regular of order $d > 0$ at $(0, 0)$. Set $c_i(x) := (\partial^i G / \partial y^i)(x, 0)$, $0 \leq i \leq d$. Assume that $c_i(x) = x^{\alpha(d-i)} c_i^*(x)$, for $0 \leq i < d$, where $\alpha \in \mathbb{N}^n$ and where $c_k^*(0) \neq 0$ for some $k < d$. Then $H(x, y) := x^{-\alpha d} G(x, x^\alpha y)$ is a quasianalytic function in a neighbourhood of $(0, 0)$, and H is y -regular of order at most k at $(0, 0)$.*

Proof. Again by (4.1), H is a quasianalytic function. Let k be the smallest integer $< d$ such that $c_k^*(0) \neq 0$. It follows from (4.3) and (4.4) that H is y -regular of order k at $(0, 0)$. \square

5. PROOF OF THE MAIN THEOREM

Proof of Theorem 1.2. The “if” direction is clear. We will prove “only if”. The problem is local, so we work in a neighbourhood of fixed $x_0 \in U$.

By assumption, there is a quasianalytic function $G : U \times (a, b) \rightarrow \mathbb{R}$, not identically zero, and vanishing on the graph of f . We will first check that we can assume, without loss of generality, that $G(x_0, \cdot)$ is not identically zero.

Let $\omega_i := (\partial^i G / \partial y^i)(x, f(x_0))$, $i \in \mathbb{N}$. Apply Theorem 3.1 to the ideal sheaf generated by the functions ω_i , $i \in \mathbb{N}$. This provides a composite of admissible blowings-up $\pi : U' \rightarrow U$ such that, near any point of $z \in \pi^{-1}(x_0)$, if \mathcal{J}_z denotes the ideal generated by the restrictions of the functions $\omega_i \circ \pi$, $i \in \mathbb{N}$, then, up to a local coordinate system at z , \mathcal{J}_z is generated by a monomial x^θ , $\theta \in \mathbb{N}^n$. This implies that every ω_i is divisible by x^θ . We claim it also follows that

$$\tilde{G}(x, y) := x^{-\theta} G(\pi(x), y)$$

is a quasianalytic function on $U' \times (a, b)$.

To see this, take $i \leq n$ such that $\theta_i \neq 0$, where $\theta = (\theta_1, \dots, \theta_n)$. It is enough to show that $G(\pi(x), y)$ is divisible by x_i , since we can iterate the argument θ_i times for each i . For any fixed x such that $x_i = 0$, the function $\hat{G}(y) := G(\pi(x), y)$ is quasianalytic and satisfies

$$\frac{d^j \hat{G}}{dy^j}(f(x_0)) = \frac{\partial^j G}{\partial y^j}(\pi(x), f(x_0)) = 0,$$

for every $j \in \mathbb{N}$. Consequently, by axiom (3), \hat{G} is identically zero. This implies that $G(\pi(x), y)$ is zero on the hyperplane $x_i = 0$, so that, by axiom (1), $G(\pi(x), y)$ is divisible by x_i , as required.

Since the ideal \mathcal{J}_z is generated by x^θ , there is an integer d such that $\omega_d \circ \pi$ coincides with x^θ up to multiplication by a local unit. Thus, for any $z \in \pi^{-1}(x_0)$, there is d such that

$$\frac{\partial^d \tilde{G}}{\partial y^d}(z, f(x_0)) \neq 0.$$

Since we can work both locally and up to a locally finite modification, we will therefore assume that, near x_0 , the function f is a root of a quasianalytic function $G(x, y)$ which is y -regular of order d at $(x_0, f(x_0))$.

Arguing by induction on d , we can assume that the main result is true for every arc-quasianalytic function which is a root of a quasianalytic function that is y -regular of order $d' < d$.

Since G is y -regular of order d , the equation

$$(5.1) \quad \frac{\partial^{d-1} G}{\partial y^{d-1}}(x, y) = 0,$$

has nonvanishing y -derivative at $(x_0, f(x_0))$. Therefore, by the implicit function theorem (axiom (2) and Remark 2.1) (5.1) implicitly defines a quasianalytic function $\varphi : U \rightarrow \mathbb{R}$ in a neighborhood of x_0 (which we continue to call U), such that $\varphi(x_0) = f(x_0)$.

Set

$$c_i(x) := \frac{\partial^i G}{\partial y^i}(x, \varphi(x)), \quad x \in U,$$

and apply Theorem 3.1 to the ideal sheaf \mathcal{I} generated by the functions $c_i^{d/(d-i)}$, $i < d-1$. This provides a composite of blowings-up after which we can assume that \mathcal{I} is generated by a monomial x^α , $\alpha \in \mathbb{N}^n$; i.e., we can assume that $c_i^{d/(d-i)} = x^\alpha \cdot c_i^*(x)$, $i = 0, \dots, d-2$, where $c_i^*(x)$ is a unit, for some i . Without loss of generality, we can also assume that $(x_0, f(x_0)) = (0, 0)$.

For simplicity, write

$$g(x) := f(x) - \varphi(x);$$

then $g(0) = 0$. If $\mathcal{I} = (0)$, then $g \equiv 0$, which means that $f \equiv \varphi$, and the result is clear. Otherwise, consider the function

$$g_1(x) := x^{-\frac{\alpha}{d!}} \cdot g(x),$$

for $x = (x_1, \dots, x_n)$ sufficiently close to 0 in $\{x : x^\alpha \neq 0\}$. Note that g_1 is a root of the function

$$G_1(x, y) := x^{-\frac{\alpha}{(d-1)!}} \cdot G(x, x^{\frac{\alpha}{d!}} y + \varphi(x)),$$

defined for $x^\alpha \neq 0$.

We first check that g_1 is bounded. Taking the Taylor expansion of $y \mapsto G(x, y + \varphi(x))$ and evaluating it at $g(x)$, we see that

$$(5.2) \quad \sum_{i=0}^{d-1} \left| (c_i^*)^{\frac{d-i}{d!}}(x) \cdot x^{\frac{\alpha(d-i)}{d!}} \cdot \frac{g(x)^i}{i!} \right| \geq \left| \rho(x, g(x)) g(x)^d \right|,$$

where $\rho(x, t)$ is a \mathcal{C}^∞ function that does not vanish at $(0, 0)$. Assume that g_1 is not bounded. Then there is a sequence x_ν tending to zero such that

$$|g(x_\nu)| \gg |x_\nu^{\frac{\alpha}{d!}}|,$$

and therefore

$$|g(x_\nu)|^d \gg |x_\nu^{\frac{\alpha(d-i)}{d!}}| \cdot |g(x_\nu)|^i,$$

for any $i < d$. This contradicts (5.2); therefore g_1 is bounded.

By Lemma 4.5, the function

$$G_1(x^{d!}, y) = x^{-\alpha d} \cdot G(x^{d!}, x^\alpha y + \varphi(x^{d!}))$$

is quasianalytic and y -regular of order at most $(d-1)$ at any point of the y -axis. This shows that G_1 extends to a continuous function on the quadrant $\{(x, y) : x_1 \geq 0, \dots, x_n \geq 0\}$. More generally, given any $p \in \{0, 1\}^n$, by the same argument (applying Lemma 4.5 to the function $G_1((-1)^{p_1} x_1^{d!}, \dots, (-1)^{p_n} x_n^{d!}, y)$), we see that G_1 is continuous on $U^p \times [-M, M]$, if U is a sufficiently small neighborhood of the origin in \mathbb{R}^n and $M > 0$. Therefore, by Lemma 4.3, $|g_1|$ extends continuously to the origin.

The problem is that the function g_1 might not be arc-quasianalytic since $\alpha/d!$ *a priori* is not an element of \mathbb{N}^n . If g_1 does not tend to zero at the origin, we will see that $\alpha/d!$ is necessarily an integer, and then G_1 is quasianalytic. (This is Case I below.) The most difficult case is when g_1 tends to zero at the origin (Case II). In this case, we will see that we can iterate the preceding argument finitely many times, and find G_2, \dots, G_k and g_2, \dots, g_k , until $\lim_{x \rightarrow 0} |g_k(x)| \neq 0$. The order of y -regularity of the G_j will be strictly decreasing, forcing the process to end after finitely many steps. The construction of the successive G_j will be done by a method very similar to that above; nevertheless, for technical reasons, we have to replace Lemma 4.5 with Lemma 4.6.

We thus distinguish two cases.

Case I. $\lim_{x \rightarrow 0} |g_1(x)| \neq 0$. By construction,

$$(5.3) \quad g_1(x) = x^{-\gamma} (f(x) - \varphi(x)),$$

for some $\gamma \in \mathbb{Q}_+^n$. We claim that $\gamma \in \mathbb{N}^n$.

To see this, consider the curve $\beta(t) = (t, \sigma, \dots, \sigma)$ where σ is a sufficiently small positive real number. Since φ is quasianalytic and f is arc-quasianalytic, $(f - \varphi)(\beta(t))$ is a quasianalytic function. Set $\gamma = (\gamma_1, \dots, \gamma_n)$. Since $|g_1|$ is continuous and does not vanish at 0, then, by (5.3), γ_1 equals the exponent of the first term of the Taylor expansion of the arc $(f - \varphi)(\beta(t))$. Thus γ_1 is an integer. Of course, we can repeat the argument for every γ_j .

Now, by Lemma 4.2, G_1 is quasianalytic (since G_1 is bounded on the nonnegative quadrant). Moreover, by Lemma 4.3, g_1 extends continuously to an arc-quasianalytic function in a neighbourhood of the origin. Therefore, g_1 is an arc-quasianalytic root of a quasianalytic function which is y -regular of order $i < d$ in a neighbourhood of 0. By induction on the order of

regularity, g_1 becomes quasianalytic after a local modification. By (5.3), therefore so does f , as required.

Case II. $\lim_{x \rightarrow 0} g_1(x) = 0$. We can assume that g is not identically zero since otherwise the result is clear. Case II ultimately reduces to Case I after iterating an argument similar to that used to construct G_1 and g_1 . Given $u \in \mathbb{N}$ and $x = (x_1, \dots, x_n)$, we set $x^u := (x_1^u, \dots, x_n^u)$. We first prove the following.

Claim 5.1. *Let $G : W \times (a, b) \rightarrow \mathbb{R}$ be a quasianalytic function (not identically zero), where W is a neighbourhood of 0 in \mathbb{R}^n , and let $g : W \rightarrow (a, b)$ be an arc-quasianalytic function such that $G(x, g(x)) \equiv 0$. Let $\pi_1 : V \rightarrow W$ be a local modification, and let $A : V \rightarrow \mathbb{R}$ be a unit (i.e., a nowhere vanishing quasianalytic function).*

Given $\alpha_1, \beta_1 \in \mathbb{Q}_+^n$, set

$$G_1(x, y) := x^{-\beta_1} G(\pi_1(x), A(x) \cdot x^{\alpha_1} y),$$

and

$$g_1(x) := \frac{x^{-\alpha_1}}{A(x)} \cdot g(\pi_1(x)),$$

(so that $g_1(x)$ is a root of $G_1(x, y)$).

Assume that α_1 and β_1 are such that $G_1(x^u, y)$ is quasianalytic and y -regular of order at most $i > 0$ at all $(x, g_1(x))$, $x \in V$, for some positive integer u . Assume also that $\lim_{x \rightarrow 0} g_1(x) = 0$.

Then there exists a locally finite covering by modifications such that, for each member $\pi_2 : U \rightarrow W$ of this covering, where U is a neighbourhood of the origin and $\pi_2(0) = 0$, there are $\alpha_2, \beta_2 \in \mathbb{Q}_+^n$ and a unit $B : U \rightarrow \mathbb{R}$ such that:

(1) *If*

$$(5.4) \quad G_2(x, y) := x^{-\beta_2} \cdot G(\pi_2(x), B(x) \cdot x^{\alpha_2} y),$$

then, for a suitable positive integer u' , $G_2(x^{u'}, y)$ is a quasianalytic function which is y -regular of order $i' < i$ at any $(a, 0)$, $a \in \pi_2^{-1}(0)$.

(2) *The function*

$$g_2(x) := \frac{x^{-\alpha_2}}{B(x)} \cdot g(\pi_2(x))$$

(which is a root of $G_2(x, y)$) is bounded on $U \setminus \{x^{\alpha_2} = 0\}$.

Proof of Claim 5.1. Note that the hypotheses of the claim are preserved by a small translation $(x, y) \mapsto (x + x_0, y)$, $x_0 \in \pi_1^{-1}(0)$. Therefore, we can focus nearby a point of V and assume it is the origin (and we can also assume $\lim_{x \rightarrow 0} g_1(x) = 0$, since otherwise we are in Case I).

We will show that each of the functions

$$\lambda_j(x) := \frac{\partial^j G_1}{\partial y^j}(x, 0)^u, \quad j = 0, \dots, i - 1,$$

is quasianalytic, for u as in the hypotheses of the claim. Indeed, λ_j coincides (up to multiplication by a unit) with

$$x^{u(j\alpha_1 - \beta_1)} \cdot \frac{\partial^j G}{\partial y^j}(\pi_1(x), 0)^u.$$

Since $(\partial^j G_1 / \partial y^j)(x^u, 0)$ is quasianalytic, the function

$$x^{u(j\alpha_1 - \beta_1)} \cdot \frac{\partial^j G}{\partial y^j}(\pi_1(x^u), 0)$$

is bounded near 0. Consequently, in some neighbourhood of 0,

$$\left| \frac{\partial^j G}{\partial y^j}(\pi_1(x^u), 0) \right| \leq C \left| x^{u(\beta_1 - j\alpha_1)} \right|,$$

where C is a positive constant. Substituting $x^{1/u}$ for x in this inequality and raising both sides to the power u , we get

$$\left| \frac{\partial^j G}{\partial y^j}(\pi_1(x), 0)^u \right| \leq C^u \left| x^{u(\beta_1 - j\alpha_1)} \right|,$$

for x near zero in the quadrant $\{x_1 \geq 0, \dots, x_n \geq 0\}$. By Lemma 4.2, the latter implies that λ_j is quasianalytic.

Now let \mathcal{I} denote the ideal sheaf generated by the $\lambda_j^{i!/(i-j)}$, $j < i$, and apply Theorem 3.1 to the ideal sheaf \mathcal{J} given by the product of \mathcal{I} and all $\lambda_j^{i!/(i-j)}$, $j < i$. The theorem provides a composite of blowings-up $\pi : U \rightarrow V$ such that $\pi^* \mathcal{J}$ is a normal crossings divisor; therefore, $\pi^* \mathcal{I}$ is a normal crossings divisor and also each $\lambda_j^{i!/(i-j)}$, $j < i$, becomes a monomial times a unit in suitable local coordinates. We can assume also that the components of π are monomials times units. Let $\pi_2 := \pi_1 \circ \pi$.

Since the components of π are normal crossings, we have

$$G_1(\pi(x), y) = A''(x) x^{-\beta'_1} G(\pi_2(x), A'(x) \cdot x^{\alpha'_1} y),$$

for some $\alpha'_1, \beta'_1 \in \mathbb{Q}_+^n$ and some units A', A'' . It follows from Lemma 4.2 that $G_1(\pi(x^u), y)$ is quasianalytic.

Up to multiplication by a unit,

$$\frac{\partial^j G_1}{\partial y^j}(\pi(x^u), 0) = \lambda_j \circ \pi(x).$$

Hence the hypotheses of Lemma 4.6 are satisfied by the quasianalytic function $G_1(\pi(x^u), y)$. By Lemma 4.6, therefore, there are $\alpha_2, \beta_2 \in \mathbb{Q}_+$, a unit $B : U \rightarrow \mathbb{R}$, and a positive integer u' , such that, if G_2 is defined as in (5.4), then the function $G_2(x^{u'}, y)$ is quasianalytic and y -regular of order at most $i' < i$ at $(0, 0)$. This gives (1).

The argument used in (5.2) and immediately following for g_1 clearly also applies to show that g_2 is bounded, giving (2). This completes the proof of Claim 5.1. \square

We can now finish the proof of Case II and therefore of the theorem. Note that G_1 and g_1 satisfy the assumptions of Claim 5.1. We can therefore apply the claim to get G_2 and g_2 , and show that $|g_2(x)|$ extends continuously to the points where $x^\alpha = 0$, by the same argument used for g_1 .

If $\lim_{x \rightarrow 0} |g_2(x)| \neq 0$, then we are done, according to Case I. Otherwise, we can apply Claim 5.1 to G_2 and g_2 . Iterating the argument, we get two sequences of functions G_k and g_k , $k = 1, \dots, l$, such that, for every k , $\lim_{x \rightarrow 0} g_k(x) = 0$ and $G_k(x^u, y)$ is y -regular of order i_k at $(0, 0)$, for suitable u , and i_2, \dots, i_l is a strictly decreasing sequence of positive integers. Of course, such a sequence cannot continue indefinitely, so that $\lim_{x \rightarrow 0} |g_l(x)| \neq 0$, for some l . (If G_l is y -regular of order 0 at $(0, 0)$, then $G_l(0, 0) \neq 0$, so that $\lim_{x \rightarrow 0} |g_l(x)| \neq 0$.) In other words, we are eventually in Case I. \square

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